

UPPER BEURLING DENSITY OF SYSTEMS FORMED BY TRANSLATES OF FINITE SETS OF ELEMENTS IN $L^p(\mathbb{R}^d)$

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ABSTRACT. In this paper, we prove that if a finite disjoint union of translates $\bigcup_{k=1}^n \{f_k(x - \gamma)\}_{\gamma \in \Gamma_k}$ in $L^p(\mathbb{R}^d)$ ($1 < p < \infty$) is a p' -Bessel sequence for some $1 < p' < \infty$, then the disjoint union $\Gamma = \bigcup_{k=1}^n \Gamma_k$ has finite upper Beurling density, and that if $\bigcup_{k=1}^n \{f_k(x - \gamma)\}_{\gamma \in \Gamma_k}$ is a (C_q) -system with $1/p + 1/q = 1$, then Γ has infinite upper Beurling density. Thus, no finite disjoint union of translates in $L^p(\mathbb{R}^d)$ can form a p' -Bessel (C_q) -system for any $1 < p' < \infty$. Furthermore, by using techniques from the geometry of Banach spaces, we obtain that, for $1 < p \leq 2$, no finite disjoint union of translates in $L^p(\mathbb{R}^d)$ can form an unconditional basis.

1. INTRODUCTION

Given $1 < p < \infty$ and $\gamma \in \mathbb{R}^d$, we define the translation operator T_γ on $L^p(\mathbb{R}^d)$ by $(T_\gamma f)(x) = f(x - \gamma)$ for all $x \in \mathbb{R}^d$. If $\Gamma \subset \mathbb{R}^d$, then the collection of translations of $f \in L^p(\mathbb{R}^d)$ along Γ is defined to be $T_p(f, \Gamma) = \{T_\gamma f\}_{\gamma \in \Gamma}$. Our main focus shall be on the upper Beurling density of such Γ , the disjoint union $\bigcup_{k=1}^n \Gamma_k$, given that $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ has some additional structure in $L^p(\mathbb{R}^d)$. The “additional structure” takes two forms: $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a p' -Bessel sequence or is a (C_q) -system.

The nature of $T_p(f, \Gamma)$ has been studied in a number of papers [21, 2, 10, 19], mainly using techniques of harmonic analysis. Our techniques will come partially from the geometry of Banach spaces. Recall that, in 1992, Olson and Zalik [20] proved that there do not exist any Riesz bases for $L^2(\mathbb{R})$ generated by $T_2(f, \Gamma)$. Then Christensen [6] conjectured that there are no frames for $L^2(\mathbb{R})$ of the form $\bigcup_{k=1}^n T_2(f_k, \Gamma_k)$. In 1999, Christensen, Deng and Heil [11] proved this conjecture by studying density of frames. For more density theorems, please see the research survey [12]. Recently, Odell, Sari, Schlumprecht and Zheng [18] used techniques largely from the geometry of Banach spaces to consider the closed subspace of $L^p(\mathbb{R})$ generated by translates of one element in $L^p(\mathbb{R})$.

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In Section 2, we extend the concept of (C_q) -system from Hilbert spaces to reflexive Banach spaces and give our basic Lemma 3.3 and examples in $L^p(\mathbb{R}^d)$.

In section 3, by using techniques in [11, 18], we prove that if $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ in $L^p(\mathbb{R}^d)$ ($1 < p < \infty$) is a p' -Bessel sequence for some $1 < p' < \infty$, then the disjoint union $\Gamma = \bigcup_{k=1}^n \Gamma_k$ has finite upper Beurling density, and that if $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a (C_q) -system with $1/p + 1/q = 1$, then Γ has infinite upper Beurling density. Thus, no collection $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ of pure translates can form a p' -Bessel (C_q) -system in $L^p(\mathbb{R}^d)$ for any $1 < p' < \infty$. This extends the Christensen/Deng/Heil density result in [11] from classical (Hilbert) frames in $L^2(\mathbb{R}^d)$ to more general p' -Bessel (C_q) -systems in $L^p(\mathbb{R}^d)$.

In the last section, by using techniques from the geometry of Banach spaces, we obtain that there is no unconditional basis of $L^p(\mathbb{R}^d)$ of the form $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ for $1 < p \leq 2$. It partially extends the latest work [18] on uniformly separated translates of one element in $L^p(\mathbb{R})$. The extension is to higher dimensions, to multiple generating functions, and to completely arbitrary sets of translates.

2. PRELIMINARIES AND NOTATION

In 2001, Aldroubi, Sun and Tang [3] introduced the concept of p -frame in $L^p(\mathbb{R})$, which is a generalization of classical (Hilbert) frames [9, 4, 7] and can be naturally extended to Banach spaces [8, 5].

Definition 2.1. Let X be a separable Banach space and $1 < p < \infty$. A family $\{f_k\}_{k=1}^\infty \subset X$ is a p -frame for X^* if there exist constants $A, B > 0$ such that

$$A\|h\|^p \leq \sum_{k=1}^{\infty} |\langle h, f_k \rangle|^p \leq B\|h\|^p \quad \text{for all } h \in X^*.$$

The number A and B are called the lower and upper p -frame bounds. The sentence $\{f_k\}_{k=1}^\infty$ is a p -Bessel sequence if the right-hand side inequality holds. We say that $\{f_k\}_{k=1}^\infty$ is a Bessel sequence if it is a 2-Bessel sequence.

In 2007, S. Nitzan and A. Olevskii introduced the concept of (C_q) -system in Hilbert spaces [15, 16, 17]. It is a weaker form of the frame-type condition, which is a relaxed version of this inequality:

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \quad \text{for all } f \in \mathcal{H}.$$

Now we extend this useful definition of (C_q) -system to reflexive Banach spaces.

Definition 2.2. Let X be a separable reflexive Banach space and $1 < q < \infty$ be a fixed number. We say that a sequence of $\{f_k\}_{k=1}^\infty \subset X$ is a (C_q) -system in X with constant $C > 0$ (complete with ℓ_q control over the coefficients) if for every $f \in X$ and $\varepsilon > 0$, there exists a linear combination $g = \sum a_k f_k$ such that

$$\|f - g\|_X < \varepsilon \quad \text{and} \quad \left(\sum |a_n|^q \right)^{1/q} \leq C\|f\|_X, \quad (2.1)$$

where $C = C(q)$ is a positive constant not depending on f .

Remark 2.3. By Proposition 4.2 in Section 4, given $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, we have that: if $1 < p \leq 2$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a q -Bessel (C_2) -system; if $2 \leq p < \infty$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a Bessel (C_p) -system.

We define some types of sequences in \mathbb{R}^d and upper Beurling density [11, 7].

Definition 2.4. Let $\Gamma = \{\gamma_i\}_{i \in I} \subset \mathbb{R}^d$.

- (i) A point $\gamma \in \mathbb{R}^d$ is an *accumulation point* for Γ if every open ball in \mathbb{R}^d centered at γ contains infinitely many γ_k .
- (ii) Γ is δ -uniformly separated if $\delta = \inf_{i \neq j} |\gamma_i - \gamma_j| > 0$. The number δ is the *separation constant*.
- (iii) Γ is *relatively uniformly separated* if it is a finite union of uniformly separated sequences Γ_k . That is to say that I can be partitioned into finite disjoint sets I_1, \dots, I_n such that each sequence $\Gamma_k = \{\gamma_i\}_{i \in I_k}$ is δ_k -uniformly separated for some $\delta_k > 0$.

For $h > 0$ and $x \in \mathbb{R}^d$, we define cube $Q_h(x)$ by

$$Q_h(x) = \prod_{i=1}^d [x_i - h/2, x_i + h/2), \quad \text{where } x = (x_1, \dots, x_d).$$

When $x = 0$, we use Q_h instead of $Q_h(0)$ for simplicity. Let $\Gamma = \{\gamma_i\}_{i \in I} \subset \mathbb{R}^d$. For each $h > 0$, let $\nu_\Gamma^+(h)$ denote the largest number of points from Γ that lie in any cube $Q_h(x)$, i.e.,

$$\nu_\Gamma^+(h) = \sup_{x \in \mathbb{R}^d} \#(\Gamma \cap Q_h(x)).$$

The *upper Beurling density* of Γ is defined by

$$D^+(\Gamma) = \limsup_{h \rightarrow \infty} \frac{\nu_\Gamma^+(h)}{h^d}.$$

Lemma 2.5. Let $\Gamma = \{\gamma_i\}_{i \in I}$ be a sequence in \mathbb{R}^d . Then the following statements are equivalent.

- (i) $D^+(\Gamma) < \infty$.
- (ii) Γ is relatively uniformly separated.
- (iii) For some (and therefore every) $h > 0$, there is a natural number N_h such that each cube $Q_h(hn)$, $n \in \mathbb{Z}^d$, contains at most N_h points from Λ . That is,

$$N_h = \sup_{n \in \mathbb{Z}^d} \#(\Lambda \cap Q_h(hn)) < \infty.$$

3. MAIN RESULTS

First we need the following basic lemma.

Lemma 3.1. Let Γ be a sequence in \mathbb{R}^d , and $1 < p, q < \infty$ with $1/p + 1/q = 1$. Assume that $f \in L^p(\mathbb{R}^d)$, $\tilde{f} \in L^q(\mathbb{R}^d)$ and $\langle f, \tilde{f} \rangle \neq 0$. If Γ is not relatively uniformly separated, then, for any $0 < \varepsilon < |\langle f, \tilde{f} \rangle|$, we have

$$\sup_{\beta \in \mathbb{R}^d} \#\{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f} \rangle| > \varepsilon\} = \infty.$$

Proof. Consider the function $x \mapsto \langle T_x f, \tilde{f} \rangle$ for all $x \in \mathbb{R}^d$. Since the function is continuous, for any $0 < \varepsilon < |\langle f, \tilde{f} \rangle|$ there is a cube Q_h for some $h > 0$ such that

$$\inf_{x \in Q_h} |\langle T_x f, \tilde{f} \rangle| > \varepsilon.$$

Consider an arbitrary $N \in \mathbb{N}$, by Lemma 2.5, there is a cube $Q_h(\beta)$ for some $\beta \in \mathbb{R}^d$, which contains at least N elements from Γ . Then for any $\gamma \in Q_h(\beta)$, $\gamma - \beta \in Q_h$, we have

$$|\langle T_\gamma f, T_\beta \tilde{f} \rangle| = |\langle T_{\gamma-\beta} f, \tilde{f} \rangle| > \varepsilon.$$

It follows that

$$\#\{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f} \rangle| > \varepsilon\} \geq \#(\Gamma \cap Q_h(\beta)) \geq N.$$

Since $N \in \mathbb{N}$ is arbitrary, the conclusion follows. \square

For translate of one element, we get the following result.

Proposition 3.2. *Let $1 < p < \infty$, f be a nonzero function in $L^p(\mathbb{R}^d)$, and Γ be a sequence in \mathbb{R}^d . If $T_p(f, \Gamma)$ is a p' -Bessel sequence for some $1 < p' < \infty$, then Γ is relatively uniformly separated.*

Proof. Assume that Γ is not relatively uniformly separated. Then for any $N \in \mathbb{N}$, choose ε such that $0 < \varepsilon < \|f\|_p$. By Hahn-Banach Theorem, there is an $\tilde{f} \in L^q(\mathbb{R}^d)$ with $\|\tilde{f}\|_q = 1$ such that $\langle f, \tilde{f} \rangle = \|f\|_p > \varepsilon$. Then, by Lemma 3.1, there exists $\beta \in \mathbb{R}^d$ such that

$$\#\{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f} \rangle| > \varepsilon\} \geq N.$$

Let $\Gamma_N = \{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f} \rangle| > \varepsilon\}$. Then, we have

$$\sum_{\gamma \in \Gamma} |\langle T_\gamma f, T_\beta \tilde{f} \rangle|^{p'} \geq \sum_{\gamma \in \Gamma_N} |\langle T_\gamma f, T_\beta \tilde{f} \rangle|^{p'} > N\varepsilon^{p'}.$$

Since $N \in \mathbb{N}$ is arbitrary and $\|T_\beta \tilde{f}\|_q = \|\tilde{f}\|_q$ is fixed, $T_p(f, \Gamma)$ is not a p' -Bessel sequence, which leads to a contradiction. Thus Γ is relatively uniformly separated. \square

The following equivalent form extends Lemma 1 in [15] by using a standard duality argument in Banach spaces.

Lemma 3.3. *Let X be a separable reflexive Banach space and $1 < p, q < \infty$ with $1/p + 1/q = 1$. A system $\{f_n\} \subset X$ is a (C_q) -system in X with constant $K > 0$ if and only if*

$$\frac{1}{K} \|h\| \leq \left(\sum_{n=1}^{\infty} |\langle h, f_n \rangle|^p \right)^{1/p} \quad \text{for all } h \in X^*.$$

Proof. For sufficiency, suppose that $\{f_n\}$ is not a (C_q) -system in X with constant $K > 0$. Let

$$A := \left\{ g = \sum a_n f_n : \left(\sum |a_n|^q \right)^{1/q} \leq K \right\}$$

be the set of finite linear combination and \mathcal{C} be the closure of A in X . It is easy to prove that \mathcal{C} is a closed convex subset of X . By assumption, \mathcal{C} does not contain

the closed unit ball B of X . That is, there exists an $f \in X$ with $\|f\| \leq 1$, and f is not in \mathcal{C} . By the Hahn-Banach theorem, there is an $h \in X^*$ such that $|\langle h, f \rangle| = 1$ and $\sup_{g \in \mathcal{C}} |\langle h, g \rangle| < 1$. Hence, for sufficiently small $\varepsilon > 0$, we have $\sup_{g \in \mathcal{C}} |\langle h, g \rangle| < 1 - \varepsilon$. This implies for any $M \in \mathbb{N}$,

$$\begin{aligned} \left(\sum_{n=1}^M |\langle h, f_n \rangle|^p \right)^{1/p} &= \sup_{(\sum_{n=1}^M |\alpha_n|^q)^{1/q} \leq 1} \left| \sum_{n=1}^M \langle h, f_n \rangle \alpha_n \right| \\ &= \frac{1}{K} \sup_{(\sum_{n=1}^M |\alpha_n|^q)^{1/q} \leq K} \left| \langle h, \sum_{n=1}^M \alpha_n f_n \rangle \right| \\ &= \frac{1}{K} \sup_{g \in \mathcal{C}} |\langle h, g \rangle| \\ &< \frac{1}{K} (1 - \varepsilon). \end{aligned}$$

By the arbitrary of M , we have

$$\left(\sum_{n=1}^{\infty} |\langle h, f_n \rangle|^p \right)^{1/p} \leq \frac{1}{K} (1 - \varepsilon) < \frac{1}{K} = \frac{1}{K} |\langle h, f \rangle| \leq \frac{1}{K} \|h\|,$$

which leads to a contradiction.

For necessity, let $\{f_n\}$ be a (C_q) -system with constant $K > 0$ in X . For every $h \in X^*$ and $\varepsilon > 0$, there exists an $f \in X$, $\|f\| = 1$, and $|\langle h, f \rangle| = \|h\|$. Choose a linear combination $g = \sum a_n f_n$ such that $\|f - g\| < \varepsilon$ and

$$\left(\sum |a_n|^q \right)^{1/q} \leq K \|f\| = K.$$

Then

$$\begin{aligned} \|h\| &= |\langle h, f \rangle| \\ &\leq |\langle h, f - g \rangle| + |\langle h, g \rangle| \\ &\leq \varepsilon \|h\| + \sum |a_n| |\langle h, f_n \rangle| \\ &\leq \varepsilon \|h\| + \left(\sum |a_n|^q \right)^{1/q} \left(\sum |\langle h, f_n \rangle|^p \right)^{1/p} \\ &\leq \varepsilon \|h\| + K \left(\sum |\langle h, f_n \rangle|^p \right)^{1/p}. \end{aligned}$$

That is,

$$\frac{1 - \varepsilon}{K} \|h\| \leq \left(\sum |\langle h, f_n \rangle|^p \right)^{1/p}, \quad \forall h \in X^*, \quad \forall \varepsilon > 0.$$

Since ε is arbitrarily small, take $\varepsilon \rightarrow 0$, we complete the proof. \square

The following result is elementary but very useful.

Lemma 3.4. *Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^d)$, and Γ be a sequence in \mathbb{R}^d . If Γ is relatively uniformly separated, then for all cubes $Q_h(x)$, for any $x \in \mathbb{R}^d$ and $h > 0$, we have*

$$(i) \sum_{\gamma \in \Gamma} \|\chi_{Q_h(x)} T_\gamma f\|_p^p < \infty. \quad (ii) \sum_{\gamma \in \Gamma} \|\chi_{Q_h(x)} T_\gamma f\|_p^p \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

Proof. (i) Since Γ is relatively uniformly separated, it is a disjoint finite union of δ_k -separated sequences Γ_k for $\delta_k > 0$ with $k = 1, \dots, n$. Let $\delta = \min_{1 \leq k \leq n} \delta_k > 0$ be the relatively separated constant and choose $0 < \varepsilon < \delta/\sqrt{d}$. Because any cube $Q_h(x)$ is bounded, it must be contained in $Q_{2N\varepsilon}$ for some $N \in \mathbb{N}$. Thus, it is enough to prove that $\sum_{\gamma \in \Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p < \infty$ for all $N \in \mathbb{N}$. For any $x \in \mathbb{R}^d$ and $h > 0$, let

$$Q_h^+(x) = x + \prod_{j=1}^d [0, h) = \prod_{j=1}^d [x_j, x_j + h).$$

Then

$$\begin{aligned} \sum_{\gamma \in \Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p &= \sum_{k=1}^n \sum_{\gamma \in \Gamma_k} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p \\ &= \sum_{k=1}^n \sum_{\gamma \in \Gamma_k} \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \|\chi_{Q_\varepsilon^+(\varepsilon a)} T_\gamma f\|_p^p \\ &= \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \sum_{\gamma \in \Gamma_k} \|\chi_{Q_\varepsilon^+(\varepsilon a)} T_\gamma f\|_p^p \\ &= \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \sum_{\gamma \in \Gamma_k} \|\chi_{Q_\varepsilon^+(\varepsilon a) - \gamma} f\|_p^p \\ &= \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \sum_{\gamma \in \Gamma_k} \int_{Q_\varepsilon^+(\varepsilon a) - \gamma} |f(x)|^p dx. \end{aligned}$$

Since $\text{diam}(Q_\varepsilon^+(\varepsilon a)) = \sqrt{d}\varepsilon < \delta$ for any $a \in Q_{2N} \cap \mathbb{Z}^d$, we get

$$Q_\varepsilon^+(\varepsilon a) - \gamma = Q_\varepsilon^+(\varepsilon a - \gamma)$$

are mutually disjoint for $\gamma \in \Gamma_k$. Thus

$$\begin{aligned} \sum_{\gamma \in \Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p &= \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \sum_{\gamma \in \Gamma_k} \int_{Q_\varepsilon^+(\varepsilon a) - \gamma} |f(x)|^p dx \quad (3.1) \\ &\leq \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \|f\|_p^p \\ &= n(2N)^d \|f\|_p^p \\ &< \infty. \end{aligned}$$

(ii) For each $k = 1, \dots, n$, $a \in Q_{2N} \cap \mathbb{Z}^d$ and fixed $x \in \mathbb{R}^d$, we have

$$\chi_{Q_\varepsilon^+(\varepsilon a) - \Gamma_k} |f(x)|^p \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

here $Q_\varepsilon^+(\varepsilon a) - \Gamma_k = \bigcup_{\gamma \in \Gamma_k} Q_\varepsilon^+(\varepsilon a) - \gamma$. Since

$$\chi_{Q_\varepsilon^+(\varepsilon a) - \Gamma_k} |f(x)|^p \leq |f(x)|^p,$$

by the Lebesgue Dominated Convergence Theorem, it follows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{\gamma \in \Gamma_k} \int_{Q_\varepsilon^+(\varepsilon a) - \gamma} |f(x)|^p dx &= \lim_{\varepsilon \rightarrow 0} \int_{Q_\varepsilon^+(\varepsilon a) - \Gamma_k} |f(x)|^p dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \chi_{Q_\varepsilon^+(\varepsilon a) - \Gamma_k} |f(x)|^p dx \\ &= 0. \end{aligned}$$

Thus by (3.1),

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{\gamma \in \Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p &= \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \sum_{\gamma \in \Gamma_k} \int_{Q_\varepsilon^+(\varepsilon a) - \gamma} |f(x)|^p dx \\ &= \sum_{k=1}^n \sum_{a \in Q_{2N} \cap \mathbb{Z}^d} \lim_{\varepsilon \rightarrow 0} \sum_{\gamma \in \Gamma_k} \int_{Q_\varepsilon^+(\varepsilon a) - \gamma} |f(x)|^p dx \\ &= 0. \end{aligned}$$

Thus, we obtain that if Γ is relatively uniformly separated, then for any $N \in \mathbb{N}$,

$$\lim_{\varepsilon \rightarrow 0} \sum_{\gamma \in \Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p = 0.$$

Since for all $x \in \mathbb{R}^d$, the translation $\Gamma - x = \{\gamma - x : \gamma \in \Gamma\}$ of Γ is relatively uniformly separated, then

$$\lim_{h \rightarrow 0} \sum_{\gamma \in \Gamma} \|\chi_{Q_h(x)} T_\gamma f\|_p^p = \lim_{h \rightarrow 0} \sum_{\gamma \in \Gamma} \|\chi_{Q_h} T_{\gamma-x} f\|_p^p = \lim_{h \rightarrow 0} \sum_{\gamma \in \Gamma-x} \|\chi_{Q_h} T_\gamma f\|_p^p = 0.$$

Now the conclusion follows. \square

Now we prove our main result.

Theorem 3.5. *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $n, d \in \mathbb{N}$. For each $k = 1, \dots, n$, choose a nonzero function $f_k \in L^p(\mathbb{R}^d)$ and an arbitrary sequence $\Gamma_k \subset \mathbb{R}^d$. Let Γ be the disjoint union of $\Gamma_1, \dots, \Gamma_n$.*

(i) *If for some $1 < p' < \infty$, $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a p' -Bessel sequence, then*

$$D^+(\Gamma) < \infty.$$

(ii) *If $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a (C_q) -system, then $D^+(\Gamma) = \infty$.*

In particular, there is no p' -Bessel (C_q) -system in $L^p(\mathbb{R}^d)$ of the form $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$.

Proof. (i) Suppose that, for some $1 < p' < \infty$, $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a p' -Bessel sequence. It is equivalent to that each $T_p(f_k, \Gamma_k)$ is a p' -Bessel sequence for $L^q(\mathbb{R}^d)$. Then, by Proposition 3.2, each Γ_k is relatively uniformly separated. By Lemma 2.5, Γ_k has finite upper Beurling density for each $1 \leq k \leq n$, i.e.

$D^+(\Gamma_k) < +\infty$. Then by definition we have

$$\begin{aligned}
\nu_\Gamma^+(h) &= \sup_{x \in \mathbb{R}^d} \#(\Gamma \cap Q_h(x)) \\
&= \sup_{x \in \mathbb{R}^d} \#(\cup_{k=1}^n (\Gamma_k \cap Q_h(x))) \\
&\leq \sum_{k=1}^n \sup_{x \in \mathbb{R}^d} \#(\Gamma_k \cap Q_h(x)) \\
&= \sum_{k=1}^n \nu_{\Gamma_k}^+(h).
\end{aligned}$$

It follows that

$$\begin{aligned}
D^+(\Gamma) &= \limsup_{h \rightarrow \infty} \frac{\nu_\Gamma^+(h)}{h^d} \\
&\leq \limsup_{h \rightarrow \infty} \frac{\sum_{k=1}^n \nu_{\Gamma_k}^+(h)}{h^d} \\
&\leq \sum_{k=1}^n \limsup_{h \rightarrow \infty} \frac{\nu_{\Gamma_k}^+(h)}{h^d} \\
&= \sum_{k=1}^n D^+(\Gamma_k) \\
&< +\infty.
\end{aligned} \tag{3.2}$$

Thus, Γ has finite upper Beurling density.

(ii) Since Γ is the disjoint union of sequences Γ_k , then, by formula (3.2), we have $D^+(\Gamma) < \infty$ if and only if $D^+(\Gamma_k) < \infty$ for each $k = 1, \dots, n$. Assume that Γ has finite upper Beurling density. By Lemma 2.5, we know that Γ_k is relatively uniformly separated. Now consider the cube $Q_{2h} = \prod_{j=1}^d [-h, h)$ for $h > 0$. Then

$$\begin{aligned}
\sum_{k=1}^n \sum_{\gamma \in \Gamma_k} |\langle \chi_{Q_{2h}}, T_\gamma f_k \rangle|^p &= \sum_{k=1}^n \sum_{\gamma \in \Gamma_k} |\langle \chi_{Q_{2h}}, \chi_{Q_{2h}} T_\gamma f_k \rangle|^p \\
&\leq \sum_{k=1}^n \sum_{\gamma \in \Gamma_k} \|\chi_{Q_{2h}}\|_q^p \|\chi_{Q_{2h}} T_\gamma f_k\|_p^p \\
&\leq \|\chi_{Q_{2h}}\|_q^p \sum_{k=1}^n \sum_{\gamma \in \Gamma_k} \|\chi_{Q_{2h}} T_\gamma f_k\|_p^p.
\end{aligned}$$

By Lemma 3.4, we have for each $k = 1, \dots, n$,

$$\sum_{\gamma \in \Gamma_k} \|\chi_{Q_{2h}} T_\gamma f_k\|_p^p \rightarrow 0 \text{ as } h \rightarrow 0.$$

Thus, by Lemma 3.3, it is easy to see that $\bigcup_k T(f_k, \Gamma_k)$ is not a (C_q) -system. Thus, we complete the proof. \square

Remark 3.6. (i) The result due to Christensen, Deng and Heil [11] is a special case of Theorem 3.5 for $p = p' = 2$.

(ii) As a consequence of Theorem 3.5, for no function $g \in L^p(\mathbb{R}^d)$ and no constants $a, b > 0$, $p' > 1$ can a collection of functions of the form $\{T_{na}E_{mb}g\}_{n \in \mathbb{Z}, m=1, \dots, M}$ be a p' -Bessel (C_q) -system in $L^p(\mathbb{R}^d)$, where the modulation operator E_{mb} on $L^p(\mathbb{R}^d)$ is defined by

$$(E_{mb}f)(x) = e^{2\pi imbx} f(x).$$

However, Hilbert frames of the infinite type $\{T_{na}E_{mb}g\}_{m,n \in \mathbb{Z}}$ exist in $L^2(\mathbb{R})$ (every Hilbert frame is a Bessel (C_2) -system). For more information on Gabor frames and density theorems, please see [7, 12].

4. NONEXISTENCE OF UNCONDITIONAL BASES OF TRANSLATES IN $L^p(\mathbb{R}^d)$

In this section, we will prove that there doesn't exist any unconditional basis of the form $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ in $L^p(\mathbb{R}^d)$ for $1 < p \leq 2$. We use standard Banach space notations as may be found in [13, 14]. Background material on bases, unconditional bases and such can be found there. For the benefit of those less familiar with these notions we recall some definitions and facts.

A biorthogonal system is a sequence $\{x_n, f_n\} \subset X \times X^*$ where $f_n(x_m) = \delta_{nm}$.

$\{x_n\} \subset X$ is a (Schauder) basis for X if for all $x \in X$, there exists a unique sequence of scalars $\{a_n\}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$. This is equivalent to saying that all $x_n \neq 0$, $\overline{\text{span}}\{x_n\} = X$ and for some $K < \infty$, all $m < l$ in \mathbb{N} and all scalars $\{a_n\}_{n=1}^l$,

$$\left\| \sum_{n=1}^m a_n x_n \right\| \leq K \left\| \sum_{n=1}^l a_n x_n \right\|.$$

The smallest such K is the basis constant of $\{x_n\}$.

$\{x_n\}$ is an unconditional basis for X if for all $x \in X$, there exists a unique sequence of scalars $\{a_n\}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$ and the convergence is unconditional. i.e. $x = \sum_{n=1}^{\infty} a_{\pi(n)} x_{\pi(n)}$ for all permutations π of \mathbb{N} .

If $\{x_n\}$ is an unconditional basis for the Banach space X and $\theta = \{\theta_n\}_{n=1}^{\infty}$ is a sequence of ± 1 's, define $S_{\theta} : X \rightarrow X$ by $S_{\theta}(\sum \alpha_n x_n) = \sum \theta_n \alpha_n x_n$. The supremum over all such $\|S_{\theta}\|$ is finite, and is called the unconditional constant of the basis [13].

The following lemma is easy to prove, which we leave to interested readers.

Lemma 4.1. *Let X be a separable reflexive Banach space with $\{x_n, f_n\} \subset X \times X^*$. Assume that $\{x_n, f_n\}$ is a biorthogonal system, that is, $\langle x_n, f_m \rangle = \delta_{nm}$ for $n, m \in \mathbb{N}$. Then $\{x_n\}$ is a seminormalized unconditional basis of X if and only if $\{f_n\}$ is a seminormalized unconditional basis of X^* .*

Recall the following known inequalities in L^p -space [1]. For $1 < p < \infty$, there exist constants $A_p, B_p > 0$ such that, if $\{f_k\}_{k=1}^{\infty}$ is a normalized C -unconditional basic sequence in $L^p(\mathbb{R}^d)$, then

$$(CA_p)^{-1} \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{\infty} a_k f_k \right\|_p \leq C \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}, \quad \text{if } 1 < p \leq 2, \quad (4.1)$$

$$C^{-1} \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} \leq \left\| \sum_{k=1}^{\infty} a_k f_k \right\|_p \leq C B_p \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \text{ if } 2 \leq p < \infty. \quad (4.2)$$

Proposition 4.2. *Given $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. Then*

- (i) *If $1 < p \leq 2$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a q -Bessel (C_2) -system.*
- (ii) *If $2 \leq p < \infty$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a Bessel (C_p) -system.*

Proof. Let $\{f_i\}$ be a seminormalized unconditional basis of $L^p(\mathbb{R}^d)$, and $\{\tilde{f}_i\} \subset L^q(\mathbb{R}^d)$ be the biorthogonal functionals of $\{f_i\}$. Then, by Lemma 4.1, $\{\tilde{f}_i\}$ is a seminormalized C -unconditional basis of $L^q(\mathbb{R}^d)$. Let $C_1 = \inf \|\tilde{f}_i\|_p$ and $C_2 = \sup \|\tilde{f}_i\|_q$.

We first prove (i). Since $1 < p \leq 2$, we have $2 \leq q < \infty$. By inequality (4.2), for all $\tilde{f} \in L^q(\mathbb{R}^d)$,

$$\begin{aligned} \left(\sum |\langle \tilde{f}, f_i \rangle|^q \right)^{1/q} &= \left(\sum \frac{1}{\|\tilde{f}_i\|_q^q} |\langle \tilde{f}, \|\tilde{f}_i\|_q f_i \rangle|^q \right)^{1/q} \\ &\leq \frac{1}{C_1} \left(\sum |\langle \tilde{f}, \|\tilde{f}_i\|_q f_i \rangle|^q \right)^{1/q} \\ &\leq \frac{C}{C_1} \left\| \sum \langle \tilde{f}, \|\tilde{f}_i\|_q f_i \rangle \frac{\tilde{f}_i}{\|\tilde{f}_i\|_q} \right\|_q \\ &= \frac{C}{C_1} \left\| \sum \langle \tilde{f}, f_i \rangle \tilde{f}_i \right\|_q \\ &= \frac{C}{C_1} \|\tilde{f}\|_q. \end{aligned}$$

Moreover, for the lower 2-frame bound, we have

$$\begin{aligned} \left(\sum |\langle \tilde{f}, f_i \rangle|^2 \right)^{1/2} &= \left(\sum \frac{1}{\|\tilde{f}_i\|_q^2} |\langle \tilde{f}, \|\tilde{f}_i\|_q f_i \rangle|^2 \right)^{1/2} \\ &\geq \frac{1}{C_2} \left(\sum |\langle \tilde{f}, \|\tilde{f}_i\|_q f_i \rangle|^2 \right)^{1/2} \\ &\geq \frac{1}{B_q C C_2} \left\| \sum \langle \tilde{f}, \|\tilde{f}_i\|_q f_i \rangle \frac{\tilde{f}_i}{\|\tilde{f}_i\|_q} \right\|_q \\ &= \frac{1}{B_q C C_2} \left\| \sum \langle \tilde{f}, f_i \rangle \tilde{f}_i \right\|_q \\ &= \frac{1}{B_q C C_2} \|\tilde{f}\|_q^2. \end{aligned}$$

Now we prove (ii). Similarly, by inequality (4.1), for all $\tilde{f} \in L^q(\mathbb{R}^d)$, we get that

$$\left(\sum |\langle \tilde{f}, f_i \rangle|^2 \right)^{1/2} \leq \frac{C A_q}{C_1} \|\tilde{f}\|_q.$$

For the lower q -frame bound, we have

$$\left(\sum |\langle \tilde{f}, f_i \rangle|^q\right)^{1/q} \geq \frac{1}{CC_2} \|\tilde{f}\|_q.$$

Thus, we complete the proof. \square

The following is the main result in this section.

Theorem 4.3. *Let $1 < p \leq 2$ and $n, d \in \mathbb{N}$. For each $k = 1, \dots, n$, choose a nonzero function $f_k \in L^p(\mathbb{R}^d)$ and an arbitrary sequence $\Gamma_k \subset \mathbb{R}^d$. Let Γ be the disjoint union of $\Gamma_1, \dots, \Gamma_n$. Then $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ can at most be an unconditional basis for a proper subspace of $L^p(\mathbb{R}^d)$.*

Equivalently, there is no unconditional basis of $L^p(\mathbb{R}^d)$ of the form $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$.

Proof. If $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is an unconditional basis of $L^p(\mathbb{R}^d)$, then, by Proposition 4.2, it is a q -Bessel (C_2) -system for $L^p(\mathbb{R}^d)$. Since $1 < p \leq 2$, $2 \leq q < \infty$ with $1/p + 1/q = 1$, we have $(\sum |a_n|^q)^{1/q} \leq (\sum |a_n|^2)^{1/2}$. Then, by (2.1) in Definition 2.2, $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a q -Bessel (C_q) -system for the whole $L^p(\mathbb{R}^d)$. It leads to a contradiction by Theorem 3.5. \square

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